(6) $\Gamma\left[\frac{1}{2}\right] = \sqrt{\pi}$

We Know that $\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\,dx$

Put $n = \frac{1}{2}$, $\Gamma\left[\frac{1}{2}\right] = 2\int_0^\infty e^{-x^2}\,dx$

$$= 2\int_0^\infty e^{-y^2}\,dy$$

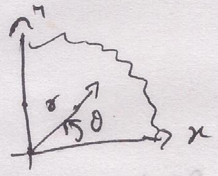
$\because\ \Gamma\left[\frac{1}{2}\right]\Gamma\left[\frac{1}{2}\right] = 2\int_0^\infty e^{-x^2}\,dx \times 2\int_0^\infty e^{-y^2}\,dy$

$$\left[\Gamma\left[\frac{1}{2}\right]\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

We use polar coordinates to evaluate the above integral

i.e $x = r\cos\theta$ & $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

Given $x : 0 \to \infty$ $y : 0 \to \infty$.



from fig we have.

$r = 0 \to \infty$ $\theta = 0 \to \pi/2$

$\therefore \left[\Gamma\left[\frac{1}{2}\right]\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$

Put $r^2 = t$ if $r = 0$ if $r = \infty$

$2r\,dr = dt$ $t = 0$ $t = \infty$.

$r\,dr = dt/2$

$$\left[\Gamma\left[\frac{1}{2}\right]\right]^2 = 4\int_{\theta=0}^{\pi/2}\int_{t=0}^\infty e^{-t}\,\frac{dt}{2}\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} -[0-1]\,d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2\theta\Big|_0^{\pi/2}$$

$$= 2\left[\pi/2 - 0\right] \Rightarrow \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \Rightarrow \boxed{\Gamma\left[\frac{1}{2}\right] = \sqrt{\pi}}$$